DISCOURSES OF FUNCTIONS – UNIVERSITY MATHEMATICS TEACHING THROUGH A COMMOGNITIVE LENS

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This paper reports on an ongoing study focusing on the teaching of functions in undergraduate courses in mathematics at three Swedish universities. In this paper excerpts from the lectures of three teachers at one university are analysed, using commognitive theory. Characteristic features of the teachers' discourses about functions are presented. Definitions are found to be the central type of narrative, while theorems and proofs are largely absent, despite the fact that the teaching is of a traditional type, often connected to the "definition-theorem-proof" format. Many routines concern the use of definitions in working with concrete examples. It is also seen that the teachers are more concerned with questions of "why" to do things than "when" to do them.

INTRODUCTION

Students' conceptions of the function concept have been extensively studied (see e.g. Harel & Dubinsky, 1992; Schwarz & Hershkowitz, 1999; Vinner & Dreyfus, 1989). In an earlier study (Viirman, Attorps & Tossavainen, in press) we looked at a small group of university mathematics students, investigating their concept images of the function concept, and making comparisons with the historical development of the concept. We worked within a theoretical framework building on the idea of concept image, as developed by Tall and Vinner (1981), and also on Sfard's (1991) theories of the process/object duality of mathematical concepts and of the three stages of concept formation. Out of this work grew a desire on my part to continue this research, but with a different focus, leading to the ongoing work of which a first report is given in the present paper. My main interest is now in the teaching, investigating how mathematics teachers at the university level work with the function concept, and what they do to promote the learning of this concept in their students. I believe that there is a need for studies of the actual practice of mathematics teaching at Swedish universities, hopefully gaining insight that might at a later stage be used to help improve university mathematics teaching.

THEORETICAL FRAMEWORK

Over the last decade or so Sfard has written extensively of the acquisition and participation metaphors as basic metaphors underlying theories of learning (cf. Sfard, 1998). Through the acquisition metaphor learning is described in terms bringing to mind the accumulation of material goods, while through the participation metaphor learning is seen as the process of becoming a member of a certain community. The framework used in our earlier study is very much based on the acquisition metaphor. The present work, however, takes a participationist view on learning.

In recent years, Sfard has developed a participationist theory of thinking (Sfard, 2008), drawing on ideas from Vygotsky and Wittgenstein. The foundational tenet of participationism is “that *patterned, collective forms of distinctly human forms of doing are developmentally prior to the activities of the individual.*” (Sfard 2008, p. 78, emph. in original) Based on this idea, Sfard defines thinking as “an individualized version of (interpersonal) communicating” (ibid, p. 81), and coins the neologism commognition in order to encapsulate both inter- and intrapersonal communication. Different types of communication are called discourses, and these discourses are in constant development, growing and increasing in complexity. Within the commognitive framework, then, learning may be defined as individualizing discourse, becoming ever more capable at communicating within the discourse, with others as well as with oneself (Sfard 2006, p. 162). This is achieved through a process of adjusting one’s discursive activities to fit the leading discourse (or, more rarely, the other way around). The unit of commognitive analysis is the discursive activity, the “patterned, collective doings” (ibid, p. 157). Hence, what I will be looking at in this study is the discourse of function, as it is manifested in the communicative practices of the teachers (and students). But what characterizes specific discourses? Sfard presents four characteristics which can be used to describe and distinguish different discourses (Sfard, 2008, p. 133ff):

word use - words specific to the discourse or common words used in discourse-specific ways

visual mediators - visual objects operated upon as a part of the discursive process. Examples from mathematical discourse could be diagrams and special symbols.

narratives - Sequences of utterances speaking of objects, relations between and/or processes upon objects, subject to endorsement or rejection within the discourse. Mathematical examples could be theorems, definitions and equations.

routines – Repetitive patterns characteristic of the discourse. Typical mathematical routines are for instance methods of proof, of performing calculations, and so on.

A more thorough presentation of the commognitive theory of mathematical discourse is beyond the scope of this paper, but a few words about routines and rules of discourse are needed. The discursive patterns are the result of processes governed by rules. Sfard distinguishes between object-level and meta-discursive rules of discourse. The former regard the properties of the objects of the discourse, while the latter govern the actions of the discursants. A routine, then, is a set of meta-rules describing a repetitive discursive action (ibid, p. 208). This set can be divided into the how and the when of the routine, determining in the first case the course of action and in the second case the situations in which action would be deemed appropriate.

So, given this, the question this study aims to answer is: What characterizes the discourses about functions presented by the teachers, primarily regarding narratives and routines?

PREVIOUS RESEARCH

The teaching of mathematics in higher education is not as well-researched as the function concept. However, the interest in this topic has increased in the last decade. An early example of research focusing on mathematics teaching at university outside of a teacher education context is the work of Burton (2004), studying professional mathematicians as learners, and possible implications for the practice of mathematics teaching at the university level. Another researcher who has done a lot of work in this area is Nardi (e.g. Nardi, 2008; Nardi, Jaworski & Hegedus 2005), investigating university mathematicians' views about the teaching of mathematics. There is also an anthology (Holton 2001), covering many aspects of university mathematics teaching. Generally, however, not that much research has focused on investigating the actual practice of mathematics teaching at the university level, although there are a number of studies, including some (e.g. Weber, 2004; Wood, Joyce, Petocz & Rodd 2007) focusing on so-called traditional mathematics instruction. In Sweden, research on the teaching of mathematics at university is rare, but one example is Bergsten (2007), discussing ways of investigating the quality of lectures in mathematics, building on a case study of one calculus lecture on limits of functions.

Since Sfard's commognitive theory is relatively recent, and still under development, not that many studies have been reported using this framework, and those that do exist tend to focus on the mathematical learning of younger children (e.g. Sfard 2001, 2007; Sfard & Lavie 2005) or on elementary mathematics, like arithmetic (Ben-Yehuda, Lavy, Linchevski & Sfard, 2005). However, very little has been published on university mathematics learning from a commognitive standpoint. The one example that I'm aware of is the work of Ryve (2006), which makes use of, but also critiques, an early version of Sfard's theory, as presented in for instance Sfard (2001), in order to investigate student interaction in problem-solving. As far as I know, there is yet no published research using commognitive theory to investigate university mathematics *teaching.*

METHOD

The empirical data in my study consists mainly of videotaped lectures and lessons given by teachers in freshman year mathematics courses at three Swedish universities, chosen for diversity – one old, large university, one more recently established, and one smaller, regional university. The teachers were then selected among those giving freshman courses on relevant topics during the time available for data collection. In two cases, this, together with the obvious fact that the teachers had to agree to participate in the study, effectively made the choice for me. At the large university, where the number of possible participants was greater, I again aimed for diversity, both in topics covered and in teaching experience. One thing all teachers in the study have in common, however, is an active interest in teaching. In the present paper, I have chosen to focus on three 45-minute excerpts from lectures given at the same university, one of the larger in Sweden. The collection and transcription of data is ongoing, and the excerpts chosen were simply the most extensive transcriptions available at the time of writing. The first excerpt is from an introductory course, mainly preparatory for calculus. The teacher (referred to as teacher A below) is a woman in her fifties, who got her doctoral degree in the 1980's, and has taught at the university for about 20 years. The second excerpt is from a course in algebra, and the teacher (B) is a male graduate student in his twenties, giving his first course as a lecturer, having earlier only served as a TA. Finally, the third excerpt is from a course in linear algebra, given by a male teacher (C) in his thirties, having recently gotten his first position following some years of post-doctoral work. The students in all three courses were first semester engineering and computer science students. The excerpts were transcribed verbatim, speech as well as the writing on the board. The transcribed lectures were then analysed, using the four characteristics of discourses described above to try to distinguish the discursive patterns characterizing the teachers' respective discourses of functions. I first analysed each lecture separately, and then looked at all three together, searching for differences and similarities. Since the unit of commognitive analysis is the discursive activity, I have intentionally chosen an outsider perspective, trying to view the enfolding discourse in as unbiased a way as possible. At the same time, I am of course aware of, and also making use of, the fact that my mathematical knowledge makes me an insider to the discourse. However, I have specifically tried to avoid making references to what is *not* present in the discourse, except in contrasting different teachers' discursive activities.

RESULTS

My focus in this paper will be on the narratives and routines characterizing the discourses of the teachers. I will therefore omit an analysis of word use and of visual mediators used. Observations regarding these will be referred to whenever they are relevant to the analysis.

A central type of narrative in all three discourses is the *definition*. These can be formal or informal in character, with informal definitions often relying heavily on metaphor. In this respect certain differences between the discourses can be seen. Teacher A consistently introduces new concepts in informal terms, and then presents a formal definition, as in the following example (all excerpts have been translated from Swedish by the author):

Teacher A maybe it's the whole of B, but maybe it's just a small part of B that actually comes out, that the machine spits out. Then we speak of the domain of *f*.

Teacher A (…) and that you could write like this if you want: *f* of *a*, the set of all *f* of *a* when you let *a* vary over all [writes: $\{f(a) : a \in A\}$]

Teacher B, however, generally starts by stating a formal definition, and then giving an informal interpretation.

Teacher B but if C is a whole subset of A, then we define f of C as simply the set of all values of the function starting in C, the set of all f of x, where x lies in C. And this leads us to our last definition [writes: $f(A)$ is called the domain of f] f of A – f of the whole set A - is called the domain of f

Teacher B so the domain is all the elements in B that the function gets to

In fact, in the case of the definition of 'function', he gives no informal interpretation at all. But there is also one case (definition of injectivity) where he introduces the concept informally before giving the formal definition. Teacher C, finally, uses a third approach. He presents the definitions formally in writing, while simultaneously explaining them verbally in an informal fashion, like this:

Teacher C [writes: Definition: $\boldsymbol{R}^m \xrightarrow{F} \boldsymbol{R}^n$] a function from R^m to R^n is called a linear map if it is linear, that is if it satisfies two conditions

Teacher C [writes: if 1) $F(v+w)$] the first one is that it respects addition in the following way: if you have two vectors v and w, and you add them, and then you apply a function F, and get vectors in R^n then that is the same thing as if you take each of the vectors, toss them into R^n and add them there.

Teacher C [writes: $=F(v) + F(w)$]

Hence, for teachers A and C, the informal definitions provide endorsement for the formal ones. As for teacher B, he often endorses his definitions through reference to a lack or need. Before giving his formal definition of function, he says that “what we haven't learned is how we connect two sets in the sense that to each element in one set we associate an element in the other set”. Interestingly enough, his definition doesn't mention the “one-valuedness” property central to the modern concept of function (to each element in the domain there is exactly one element in the range). Instead he introduces this later, endorsing it partly through another reference to need – without it we wouldn't be able to work with functions at all – and partly by referring to the metaphorical description of the function as a machine.

As for other central narratives of scholarly mathematical discourse, *theorems* and *proofs*, these are much more rare in the discourses of the three teachers. Only teacher C actually states and proves a theorem. Instead the narratives through which the discourses are developed are mainly examples illustrating specific properties. For teacher A, these examples are almost invariably given primarily by formulas. The one exception, a fairly convoluted function, used to illustrate the fact that the rule in the definition of function doesn’t require an actual method of computation, is explicitly referred to as a “silly example”. The examples given by teachers B and C are more varied, including geometrically constructed functions (rotations and projections), and functions given by tables of values.

Regarding the routines, the differences in discursive practices are more pronounced. A common feature, however, is that many routines concern the use of definitions. All three teachers present routines for checking whether a specific example satisfies a given definition. An example, from Teacher B:

Teacher B	This function is injective
Teacher B	Such a statement, bold as it is, you show by assuming that the values of the function are equal, and from that showing that this implies that x_1 and x_2 are equal

He then goes on using formal algebraic manipulation to show that the last statement follows from the first. On the other hand, in a passage, which unfortunately is too long to be quoted here, teacher C uses graphical representations of plane vectors to show that rotation in the plane by an angle $\pi/2$ satisfies the two conditions in the definition of a linear map. Having done one vector addition, he then appeals to the geometric intuition of the students:

Teacher C	yes, I have just taken that picture and moved it here through rigid rotation, so that parallelogram addition up here has to function just like that one
Teacher C	So that rotation satisfies that condition

Both examples can be seen as substantiation routines, but as we have seen, the means of substantiation are different.

Related to these routines are those using definitions to exclude non-examples.

Teacher A	if we return to this crazy example which wasn't a function, the circle
Teacher A	It wasn't a function because even if we insert something between -1 and 1, we would perhaps want that it was a function with *A* being -1 to 1, the interval, but then when we insert something which isn't 1 and not -1 [she clearly marks the origin in an already drawn picture of the unit circle in the *x*-*y*-plane] then we get two *y*-values that fit, it is different
Teacher A	So that the function doesn't give us exactly one, it gives us more than one
Teacher A	And then it isn't a function

Here, the teacher uses a graphic argument, directly showing why the circle cannot be the graph of a function. Examples of this kind of routine can be found elsewhere in her lecture, as well as in that of teacher B. They are however not found in the discourse of teacher C.

Other types of routines commonly found in the discourses of the three teachers, but taking very different forms, are construction routines. In the discourse of teacher A we can find repeated use of routines for constructing the graph of a function given an algebraic formula, and for finding the value of a function at a point given the graph. These constructions are all very sketchily outlined, however, and it is clearly stated ("we have already talked about this when we did quadratic curves") that this is something which they have been doing for some time. Also there are several instances of routines for determining the range or the largest possible domain of a function given a formula. A typical example:

Teacher A	it doesn't say on the board what V_f is. You have to look, what function is it, and then you have to try to figure out what values can come out here then
Teacher A	and in this case, there's a square here, plus 2, squares can be 0 or larger
Teacher A	and it can be any number larger than 0 this square, so it can be 2 plus something positive, so it can be any number larger than or equal to 2. So V_f in this example is the interval starting at 2 and continuing upwards

As we can see, this is something done by looking at the formula, not by for instance looking at the graph, which had been drawn just 20 minutes earlier (although it had now been erased).

Routines for determining domain and range are also present in the discourse of teacher B, and handled in pretty much the same way. A more interesting type of routine very much present in the discourse of teacher B has to do with presenting motives, and developing new mathematics. The discursive pattern can be described in the following way: OK, so now we know this, but here is something else that we don't know. What would we need in order to find this out, or be able to do this? We would need this. OK, let's define it. We have already seen one example, in the motivation given for introducing the concept of function, and here is another:

Teacher B	now that we know what a function is, then maybe one wants to illustrate it in a better way than just writing like this, like in this way you don't see, ok you know what the function does, but you don't really get any idea of what it looks like

This is followed by a description of the “function machine”, and by the definition of the graph of a function. This type of motivational pattern is not found in the discourses of the other two teachers.

The construction routines in the discourse of teacher C are mainly concerned with constructing maps from matrices, and vice versa. This is a central topic of the lecture, being the subject of the theorem which gets stated and proven. In fact, the proof is basically an instance of such a routine. There are variants using vector algebra and geometric reasoning, but both follow this basic pattern:

Teacher C	it is easy to solve that type of problem
Teacher C	if you know the images of the basis vectors under a certain linear map then you put them as columns, and that is the standard matrix of the map

A general trait of the discourse of teacher C is his frequent use of geometric reasoning. The other two teachers are much more reliant on algebraic methods.

DISCUSSION

First I want to comment on the invisibility of the students in what has been presented here. This is partly due to the fact that the excerpts are from lectures, where students tend to be less active. However, there is a certain amount of teacher-student interaction in the excerpts, some of it very interesting, but an analysis of this is beyond the scope of this paper. Here I will instead give some reflections on what I see as the character of the teachers' discourses.

All three teachers give traditional lectures, speaking and writing at the board. However, the traditional “definition-theorem-proof “ (DTP) format (Weber, 2004) is not so apparent, theorems and proofs being largely absent. There are at least two possible reasons for this. Firstly, in all three lectures, the function concept is introduced for the first time in the course, leading to a greater prevalence of definitions and examples, with theorems possibly following later. Secondly, all three courses are first-semester courses, and as such situated somewhere between elementary and advanced courses, perhaps making the DTP format less applicable.

While sharing the overall lecture format, the teachers differ in the discourses they present. These differences are obviously to a great extent predicated by differences in course content. In the discourse of teacher A, teaching to prepare for calculus, the formula is very much the preferred realization of function. This is seen in the way she introduces examples and also in her choice of words. She almost consistently speaks of curves rather than graphs, indicating that what she has in mind is continuous functions. When she does give an example of a function not given by a formula she describes it as “silly”. Teacher B, teaching algebra, uses the language and methods of formal logic in a way neither of the other teachers do. He also presents a much more general idea of what a function might be. I have already mentioned the prevalence of geometric reasoning in the discourse of teacher C (linear algebra). But there are some surprises as well. In Nardi (2008, p. 167), the mathematician (actually a composite portrait of a number of mathematicians interviewed by Nardi) claims that the graph as a realization of function is essential in analysis but meaningless in algebra. Still, teacher B devotes quite some time to the concept of graph, discussing it at a high level of detail. A possible explanation for this could be laying groundwork for the use of functions in other courses. If so, this parallel is not made explicit, something which is typical of all three lectures. Connections between different branches of mathematics are seldom made, despite the fact that functions are central to so much of mathematics. This lack is also noted in Nardi (2008, p. 167). I would also like to comment on the process/object duality (Sfard, 1991) of the function concept in the context of the discourses of the three teachers. All three primarily talk of functions in process terms. However, at the same time, they obviously view them as objects:

Teacher A — it is a function; it is the function x^2 which I move one step to the right and two steps up

Teacher C we speak of the domain of the function, where the function starts (…) and of the target set, where the function is going

It would be interesting to know what effect, if any, this might have on students' learning.

Sfard (2008) speaks of the “how” and “when” of routines. The “when” is seen as more problematic, due to the fact that “Presenting the *when* of routines, that is, constructing exhaustive lists of conditions under which given patterns tend to appear in a discourse of a given group or person, is more complicated, if not altogether unworkable.” (ibid, p. 209) For analytical purposes the questions of “how” and “when” are natural, since they are what can be observed. But in the actual discursive practices of the teachers, the question of “when” to use a certain discursive pattern is mostly addressed by discussing “why” it should be used. Perhaps a reason for this is that explaining why something is done can give more comprehensive criteria of when it should be done, avoiding the problem of presenting conditions of use.

REFERENCES

Ben-Yehuda, M., Lavy, I., Linchevski, L & Sfard, A. (2005). Doing wrong with words: What bars students' access to arithmetical discourses. *Journal for Research in Mathematics Education*, *36*(3), 176-247.

Bergsten, C. (2007). Investigating Quality of Undergraduate Mathematics Lectures. *Mathematics Education Research Journal, 19*(3), 48-72.

Burton, L. (2004). *Mathematicians as Enquirers: Learning about Learning Mathematics.* Dordrecht: Kluwer.

Harel, G. & Dubinsky, E. (Eds.). (1992). *The Concept of Function: Aspects of Epistemology and Pedagogy.* Washington, DC: Mathematical Association of America.

Holton, D. A. (Ed.) (2001). *The teaching and learning of mathematics at university level: An ICMI Study.* Dordrecht: Kluwer.

Nardi, E. (2008). *Amongst Mathematicians. Teaching and Learning Mathematics at University Level.* New York: Springer.

Nardi, E., Jaworski, B. & Hegedus, S (2005). A spectrum of pedagogical awareness for undergraduate mathematics: From ‘tricks’ to ‘techniques’. *Journal for Research in Mathematics Education, 36*, 284-316.

Ryve, A. (2006). *Approaching mathematical discourse: two analytical frameworks and their relation to problem solving interactions*. Department of Mathematics and Physics. Mälardalen University.

Schwarz, B. & Hershkowitz, R. (1999). Prototypes: Brakes or Levers in Learning the Function Concept? The Role of Computer Tools. *Journal for Research in Mathematics Education*, *30*, 362-389.

Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin. *Educational Studies in Mathematics, 22,* 1-36.

Sfard, A. (1998). On Two Metaphors for Learning and the Dangers of Choosing Just One. *Educational Researcher, 27*(2), 4-13.

Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics, 46*, 13-57.

Sfard, A. (2006). Participationist Discourse on Mathematics Learning. In J. Maasz & W. Schloeglmann (Eds.), *New Mathematics Education Research and Practice* (153-170). Rotterdam: Sense.

Sfard, A. (2007). When the Rules of Discourse Change, but Nobody Tells You: Making Sense of Mathematics Learning From a Commognitive Standpoint. *The Journal of the Learning Sciences, 16*, 565-613.

Sfard, A. (2008). *Thinking as Communicating. Human Development, the Growth of Discourses, and Mathematizing.* Cambridge, U.K.: Cambridge University Press.

Sfard, A. & Lavie, I. (2005). Why cannot children see as the same what grown-ups cannot see as different? Early numerical thinking revisited. *Cognition and Instruction*, *23*, 237-309.

Tall, D. & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with particular reference to Limits and Continuity. *Educational Studies in Mathematics, 12,* 151-169.

Viirman, O., Attorps, I. & Tossavainen, T. (in press). Different Views – Some Swedish Mathematics Students' Concept Images of the Function Concept. *Nordic Studies in Mathematics Education.*

Vinner, S. & Dreyfus, T. (1989). Images and Definitions for the Concept of Function. *Journal for Research in Mathematics Education, 20,* 356-366.

Weber, K. (2004). Traditional instruction in advanced mathematics courses: a case study of one professor's lectures and proofs in an introductory real analysis course. *Journal of Mathematical Behavior, 23*, 115-133.

Wood, L. N., Joyce, S., Petocz, P. & Rodd, M. (2007). Learning in lectures: multiple representations. *International Journal of Mathematical Education in Science and Technology, 38*, 907-915.